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Integrability of the gauged linear sigma model for $AdS_5 \times S^5$

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Abstract: Recently, a gauged linear sigma model was proposed by Berkovits and Vafa which can be used to describe the $AdS_5 \times S^5$ superstring at finite and zero radius. In this paper we show that the model is classically integrable by constructing its first non-local conserved charge and a superspace Lax "quartet". Quantum conservation of the non-local charge follows easily from superspace rules.

Keywords: AdS-CFT Correspondence, Conformal Field Models in String Theory

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Contents

1 Introduction

Over the past ten years there has been much activity in the AdS/CFT correspondence. This powerful conjecture [1] relates two different theories in different regimes. It is very difficult to prove the correspondence fully, since this would involve a complete solution of the theories on both sides. Nevertheless, we would like to see how the fundamental degrees of freedom on one side of the conjecture appear on the other side.

One particular limit which could be interesting to analyze is the limit in which the super Yang-Mills theory is free. Although we have a trivial theory on one side of the conjecture, the dynamics of the string theory side is governed by a highly interacting worldsheet. This limit is beyond the reach of perturbation theory using the Metsaev-Tseytlin $AdS_5 \times S^5$ Green-Schwarz sigma model [2] (or its pure spinor [3] version [4, 5]). Although both versions appear to be integrable two-dimensional field theories [6–8], no one has been able to use integrability to perform a non-trivial calculation which could shed light on the strongly-coupled regime. A possible approach to this problem was recently proposed by Berkovits and Vafa [9]. Using a modified version of the pure spinor action in $AdS_5 \times S^5$, they were able to define a gauged linear sigma model which is related to the usual superspace variables by a twistor-like field redefinition. The model so obtained has $N = (2,2)$ worldsheet supersymmetry, global U(2, 2|4), and local U(4) symmetry. The fact that the global symmetry group is a supergroup has important implications for the quantum theory. After integrating out the gauge degrees of freedom, one recovers the non-linear sigma model action previously obtained in [10].

Although the original motivation in [9] was to construct an action in which the zero-radius limit is reachable, the non-linear sigma model action is supposed to be equivalent to the pure spinor version for all radii. Furthermore, since it has $N = (2,2)$ worldsheet supersymmetry and space-time supersymmetry it is possible that many quantum calculations are greatly simplified. In this work we show that this is indeed the case.

A subtle point is the definition of the physical spectrum. Although Berkovits and Vafa refer to their model as an "A-model", the physical spectrum, which is supposed to be equivalent to the pure spinor version, is not the usual cohomology of an A-model since, the BRST charge of the pure spinor description is not mapped to the BRST charge of the A-model. Only the low-lying excitations, which were used in [9], should agree using the two different BRST charges.[1]

In this paper we study the classical and quantum integrability of this gauged linear sigma model. The worldsheet supersymmetry plays an important role in constraining the form of possible quantum corrections in the effective action and correlators, and space-time supersymmetry helps to prove that many of these corrections vanish. The end result is that the first non-local charge is a well-defined operator in the quantum theory and does not need renormalization. This provides further evidence that the gauged linear sigma model picture is a consistent description of the pure spinor superstring in $AdS_5 \times S^5$.

Integrability techniques are well developed on the YM side of the conjecture where the full S-matrix [12, 13] and Bethe equations, which determine the anomalous dimensions of gauge theory operators in the long operator limit, was already derived [14–16]. Also, a complete anomalous dimension function of some particular gauge theory operator which, was shown to agree with both perturbative YM [17] and string theory [18] sides, was constructed in [16]. We hope that the high number of space-time and worldsheet symmetries of this gauged linear sigma model will facilitate the implementation of such a program on the string theory side. It would be very interesting to see how the methods of [19–25] can be applied to the present case.

This paper is organized as follows. In section 2, we introduce the gauged linear sigma model proposed by Berkovits and Vafa. In section 3, we discuss its classical symmetries and find the corresponding non-local conserved charges. Section 4 is devoted to the discussion of classical integrability. In section 5, we address the question of quantum integrability of the sigma model. We conclude and discuss open problems in section 6. In the appendix we put definitions and derivations which were skipped in the main text.

[1]The topological sector of the sigma model was recently used in [11] to compute amplitudes in the open string sector of $\frac{1}{2}$-BPS operators.

2 Definition of the GSLM

The gauged linear sigma model defined by Berkovits and Vafa [9] is related to the pure spinor $AdS_5 \times S^5$ sigma model after a BRST-trivial term is added to the action. This BRST-trivial term enhances the target space symmetries and makes it possible to describe the model in terms of an $N = (2, 2)$ supersymmetric worldsheet action principle.

The resulting model resembles the old Grassmannian sigma models on $\frac{\mathrm{U}(n+m)}{\mathrm{U}(n)\times\mathrm{U}(m)}$ (see, e.g. [26]), but we replace the numerator with the supergroup $\mathrm{U}(2, 2|4)$ (and also replace one of the $\mathrm{U}(4)$s with $\mathrm{U}(2, 2)$).[2] The second fundamental difference is that, by construction, the worldsheet fields are fermionic and will have a kinetic term with two derivatives. The choice of denominator makes the model a symmetric space, in contrast with the $AdS_5 \times S^5$ sigma model which also has a Wess-Zumino term.

We begin by establishing some $N = (2, 2)$ superspace notation. Bosonic worldsheet coordinates will be denoted by $(\sigma^{\ddagger}, \sigma^{=})$ and the fermionic coordinates will be denoted by $(\kappa^+, \kappa^-, \bar\kappa^+, \bar\kappa^-)$. The covariant superderivatives are taken to be

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\kappa^+} - i\bar\kappa^+\frac{\partial}{\partial\sigma^{\ddagger}}, & \bar D_+ &= \frac{\partial}{\partial\bar\kappa^+} - i\kappa^+\frac{\partial}{\partial\sigma^{\ddagger}}, \\ D_- &= \frac{\partial}{\partial\kappa^-} - i\bar\kappa^-\frac{\partial}{\partial\sigma^{=}}, & \bar D_- &= \frac{\partial}{\partial\bar\kappa^-} - i\kappa^-\frac{\partial}{\partial\sigma^{=}}. \end{aligned} \tag{2.1}$$

They commute with the supercharges

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\kappa^+} + i\bar\kappa^+\frac{\partial}{\partial\sigma^{\ddagger}}, & \bar Q_+ &= \frac{\partial}{\partial\bar\kappa^+} + i\kappa^+\frac{\partial}{\partial\sigma^{\ddagger}}, \\ Q_- &= \frac{\partial}{\partial\kappa^-} + i\bar\kappa^-\frac{\partial}{\partial\sigma^{=}}, & \bar Q_- &= \frac{\partial}{\partial\bar\kappa^-} + i\kappa^-\frac{\partial}{\partial\sigma^{=}}, \end{aligned} \tag{2.2}$$

and satisfy the anticommutation relations

$$\{D_+, \bar D_+\} = -2i\partial_{\ddagger}\ ,\ \{D_-, \bar D_-\} = -2i\partial_{=}, \tag{2.3}$$

where $\partial_{\ddagger} = \partial/\partial\sigma^{\ddagger}$ and $\partial_{=} = \partial/\partial\sigma^{=}$. Any other graded commutator vanishes. Integration over the full superspace is defined as

$$\int d^4\kappa\ = D_+D_-\bar D_+\bar D_-\ \big|_{\kappa^+=\kappa^-=\bar\kappa^+=\bar\kappa^-=0}. \tag{2.4}$$

Analogously to the bosonic Grassmannian [26] sigma models, we introduce the basic fields $\Phi^{\Sigma}_{R}(\sigma, \kappa)$. Here Σ is a global $\mathrm{U}(2, 2|4)$ index which splits into $A = 1, \ldots, 4$ and $J = 1, \ldots, 4$, where A is a bosonic global $\mathrm{U}(2, 2)$ index, and J is a fermionic global $\mathrm{U}(4)$ index. R is a fermionic local $\mathrm{U}(4)$ index which will be gauged by introducing a gauge prepotential $V^{R}_{S}(\sigma, \kappa)$. Note that since $\mathrm{U}(2, 2|4)$ is a supergroup, and Φ^{Σ}_{R} is in its fundamental representation, Φ^{A}_{R} is a fermionic superfield and Φ^{J}_{R} is a bosonic superfield.

The superfields come in chiral/anti-chiral pairs

$$\bar D_+\Phi^{\Sigma}_{R} = \bar D_-\Phi^{\Sigma}_{R} = 0, \quad D_+\bar\Phi^{R}_{\Sigma} = D_-\bar\Phi^{R}_{\Sigma} = 0, \tag{2.5}$$

[2] For a review of supergroups, see appendix A.

which have the following expansion in terms of component fields:[3]

$$\Phi^\Sigma_R = \phi^\Sigma_R + \kappa^+ X^\Sigma_R + \kappa^- \bar{Y}^\Sigma_R + \kappa^+\kappa^- F^\Sigma_R + \cdots ,$$
$$\bar{\Phi}^R_\Sigma = \bar{\phi}^R_\Sigma + \bar{\kappa}^+ Y^R_\Sigma + \bar{\kappa}^- \bar{X}^R_\Sigma + \bar{\kappa}^+\bar{\kappa}^- F^R_\Sigma + \cdots , \tag{2.6}$$

where X^Σ_R will (after fixing an appropriate gauge) be a twistor-like combination of the $AdS_5 \times S^5$ coordinates and pure spinor ghosts, Y^R_Σ are the conjugate momenta for the twistor variables, and F^Σ_R are auxiliary fields. The higher components are not independent fields and are required only for chirality.

The prepotential for the U(4) symmetry has the following expansion in Wess-Zumino gauge:

$$V^R_S = \sigma^R_S \kappa^+\bar{\kappa}^+ + \bar{\sigma}^S_R \kappa^-\bar{\kappa}^+ + (A_{\ddagger})^R_S \kappa^+\bar{\kappa}^+ + (A_{=})^R_S \kappa^-\bar{\kappa}^- + \cdots , \tag{2.7}$$

where the ellipsis contains the gauginos and higher components. Note that in this gauge

$$e^V = 1 + V + \frac{1}{2}V^2, \tag{2.8}$$

where all terms above are matrices.[4] The prepotential has twisted-chiral field strengths given by

$$\Sigma \doteq \{\bar{\mathcal{D}}_+, \mathcal{D}_-\} = \bar{D}_+(e^{-V} D_- e^V) \ , \ \tilde{\Sigma} \doteq \{\bar{\mathcal{D}}_-, \mathcal{D}_+\} = \bar{D}_-(e^{-V} D_+ e^V), \tag{2.9}$$

where, in the gauge-chiral representation, the covariant derivatives are given by

$$\mathcal{D}_\pm = e^{-V} D_\pm e^V, \quad \bar{\mathcal{D}}_\pm = \bar{D}_\pm. \tag{2.10}$$

The above field strengths are related to the usual chiral field strength defined in four-dimensional, $N = 1$ theories by

$$W_- = \bar{\mathcal{D}}_- \Sigma, \quad W_+ = \bar{\mathcal{D}}_+ \tilde{\Sigma}. \tag{2.11}$$

Another utility of the twisted-chiral field strengths is the addition of a twisted-chiral superpotential to the model. For the present case, only a linear superpotential will be added

$$\mathcal{W}(\Sigma) = \frac{\mathbf{t}}{2}\Sigma, \quad \tilde{\mathcal{W}}(\tilde{\Sigma}) = \frac{\bar{\mathbf{t}}}{2}\tilde{\Sigma}, \tag{2.12}$$

where $\mathbf{t} = t + i\frac{\theta}{2\pi}$, t will represent the squared radius of the sigma model, and θ couples to the first Chern class of the gauge field. Unlike bosonic Grassmannian sigma models, there are no dynamical corrections to this superpotential [30].

The action for this model is given by[5]

$$S = \int d^2\sigma d^4\kappa \left[\bar{\Phi}_\Sigma e^V \Phi^\Sigma + \frac{1}{g^2}\mathrm{Tr}(\Sigma\tilde{\Sigma})\right] + \int d^2\sigma d\kappa^+ d\bar{\kappa}^- \frac{\mathbf{t}}{2}\mathrm{Tr}(\Sigma) + \int d^2\sigma d\bar{\kappa}^+ d\kappa^- \frac{\bar{\mathbf{t}}}{2}\mathrm{Tr}(\tilde{\Sigma}), \tag{2.13}$$

[3]This differs from the expansion in [9] which has the wrong component fields in the antichiral field.
[4]To avoid cumbersome notation, we sometimes omit global Σ, local R, or both indices.
[5]This type of gauged linear sigma model for Grassmannian manifolds was discussed in [31].

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where g is the coupling constant for the gauge field with dimensions of (length)$^{-1}$. Here, and in the rest of the paper, $\mathrm{Tr}(\cdot)$ denotes the trace over U(4) indices. The equations of motion for Φ_S^Σ and $\bar{\Phi}_\Sigma^S$ with arbitrary g and $\mathbf{t}$ are

$$D_+ D_-[(e^V)^R_S \Phi^\Sigma_R] = 0, \quad \bar{D}_+ \bar{D}_-[\bar{\Phi}^R_\Sigma (e^V)^S_R] = 0. \tag{2.14}$$

In the deep infra-red limit, $g \to \infty$, the equation of motion for V that follows from this action is

$$t\delta^R_S = \bar{\Phi}^T_\Sigma (e^V)^R_T \Phi^\Sigma_S, \tag{2.15}$$

whence we find that t has an interpretation as the "size" of the super-Grassmannian manifold. Another way to see this is to write the action as

$$S = \int d^2\sigma d^4\kappa \left[\bar{\Phi}^R_\Sigma \Phi^\Sigma_R + \frac{1}{2} \bar{\Phi}^R_\Sigma (V^2)^S_R \Phi^\Sigma_S + V^R_S (\bar{\Phi}^S_\Sigma \Phi^\Sigma_R - t\delta^S_R) + \cdots \right], \tag{2.16}$$

where the ellipsis denotes terms which vanish in Wess-Zumino gauge, and we set $\theta = 0$ and $g \to \infty$. We can clearly see how the familiar constraint $\bar{\Phi}^S_\Sigma \Phi^\Sigma_R = R^2 \delta^S_R$ appears with $t = R^2$: Besides being responsible for the gauge invariance, V also plays the role of the Lagrange multiplier in the $g \to \infty$ limit. It constrains the dynamical system defined by the action (2.13) to the Grassmannian and is of a different nature than the differential equation of motion (2.14). We will therefore distinguish the consequences of these two conditions by referring to equations holding due to (2.15) as *off-shell* and those holding due to (2.14) as *on-shell*.

The solution of equation (2.15) is

$$V^R_S = \delta^R_S \log t - \log(\bar{\Phi}^R_\Sigma \Phi^\Sigma_S). \tag{2.17}$$

Substituting this equation back into the action, we get a non-linear action in terms of $(\Phi, \bar{\Phi})$. Subsequently, using the U(4) gauge invariance to fix[6] $\Phi^J_R = \sqrt{t}\delta^J_R$, we obtain

$$S = t \int d^2z d^4\kappa \, \mathrm{Tr} \left[\log \left(\delta^J_K + \frac{1}{t} \bar{\Phi}^J_A \Phi^A_K \right) \right] \tag{2.18}$$

which is the usual $N = (2, 2)$ non-linear sigma model action for Grassmannian manifolds.

We close this section with some comments on the interpretation of this gauged linear sigma model. The worldsheet supersymmetry is A-twisted, which means that the components $(X^\Sigma_R, \bar{X}^R_\Sigma)$ of the (anti)chiral fields defined in equation (2.6) have conformal weight zero and the components $(\bar{Y}^\Sigma_R, Y^R_\Sigma)$ have conformal weight one.[7] However, the worldsheet operators generating the superconformal transformations are *not* the operators whose cohomology defines the physical spectrum. This fact is due to the nontrivial mapping [9] between the pure spinor variables and the variables in equation (2.6). This mapping,

[6] Although useful, this gauge fixing is not very convenient when one wants to study the relation between the GLSM and the pure spinor version [9, 30].

[7] One should be careful when talking about conformal symmetry in the present case since, as usual in gauged linear sigma models, the action is only supposed to be conformaly invariant in the infrared limit.

which involves two tensors $(\epsilon_{AB}, \epsilon_{JK})$ (in addition to those defined in appendix A) which explicitly break the $\mathrm{U}(2,2|4)$ symmetry, breaks worldsheet supersymmetry. In conclusion, although the action (2.18) is topological in the sense that it can be written in a BRST-exact form, the spectrum and correlation functions are not those of a topological theory.

3 Classical symmetries

In this section we analyze the symmetries of the action (2.13). Our goal is to verify that this two-dimensional field theory is integrable at both the classical and quantum level. Although the interpretation of the model is subtle, since it involves a field redefinition of the standard worldsheet variables, we have a well-defined field theory in two dimensions, and it is worthwhile to study its properties. Little is known about sigma models on supergroup manifolds. It was shown in [26] that the pure bosonic Grassmannian sigma model is not integrable at the quantum level but its $N = 1$ supersymmetric extension is. We would like to know the analogous statement for the present model.

When $t \neq 0$ and $g \to \infty$ we can integrate V out and get a non-linear sigma model (2.18) [10]. When $t = 0$ this procedure cannot be carried out. It would be interesting to analyze both cases, but since the latter does not appear to have a clear geometric interpretation, we will restrict our attention to the case $t \neq 0$ in this work.

Let us first analyze the local and global symmetries of equation (2.13). The $\mathrm{U}(4)$ gauge transformations are given by

$$\begin{aligned} \delta\Phi^{\Sigma}_{R} &= \delta L^{S}_{R}\Phi^{\Upsilon}_{S}, & \delta\bar{\Phi}^{R}_{\Sigma} &= (\delta L^{\dagger})^{R}_{S}\bar{\Phi}^{S}_{\Upsilon} = -\delta L^{R}_{S}\bar{\Phi}^{S}_{\Upsilon}, \\ \delta(e^{V})^{R}_{S} &= \delta L^{R}_{T}(e^{V})^{T}_{S} - (e^{V})^{R}_{T}\delta L^{T}_{S}, \\ \delta\Sigma^{R}_{S} &= \delta L^{R}_{T}\Sigma^{T}_{S} - \Sigma^{S}_{T}\delta L^{T}_{S}, \end{aligned} \tag{3.1}$$

where δL^{R}_{S} is the parameter for the $\mathrm{U}(4)$ gauge transformation. We can see more clearly the invariance of the action using matrix notation:

$$\begin{aligned} \delta\Phi^{\Sigma} &= \delta L\Phi^{\Sigma}, & \delta\bar{\Phi}_{\Sigma} &= -\bar{\Phi}_{\Sigma}\delta L, \\ \delta e^{V} &= [\delta L, e^{V}], \\ \delta\Sigma &= [\delta L, \Sigma]. \end{aligned} \tag{3.2}$$

The action (2.13) also has global $\mathrm{U}(2,2|4)$ invariance

$$\begin{aligned} \delta_{\mathrm{global}}\Phi_{R} &= \delta M\Phi_{R}, & \delta_{\mathrm{global}}\bar{\Phi}^{R} &= -\bar{\Phi}^{R}\delta M, \\ \delta_{\mathrm{global}}e^{V} &= 0, & \delta_{\mathrm{global}}\Sigma &= 0, \end{aligned} \tag{3.3}$$

where δM is the parameter for the global $\mathrm{U}(2,2|4)$ transformation. To compute the conserved current associated with this global symmetry, we promote the parameter of the transformation for Φ to a chiral superfield δM and the one for $\bar{\Phi}$ to an antichiral superfield $\delta\bar{M}$. The variation of the action is

$$\delta S = \int d^2 z d^4\kappa [-\bar{\Phi}^{S}\delta\bar{M}(e^{V})^{R}_{S}\Phi_{R} + \bar{\Phi}^{S}(e^{V})^{R}_{S}\delta M\Phi_{R}], \tag{3.4}$$

this variation is zero when $\delta M = \delta \bar{M}$, that is, when M is a constant superfield. Varying with respect to to $\delta \bar{M}$ we get

$$\frac{\delta S}{\delta \bar{M}^{\Upsilon}{}_{\Sigma}} = -(-1)^{|\Upsilon||\Sigma|} D_+ D_- (\bar{\Phi}^S_{\Upsilon} (e^V)^R_S \Phi^{\Sigma}_R), \tag{3.5}$$

so $D_+ D_- (\bar{\Phi}^S_{\Upsilon} (e^V)^R_S \Phi^{\Sigma}_R) = 0$ is the conservation law associated with the global invariance. As usual, the conservation law is only valid on-shell (eq. 2.14). We will define the corresponding gauge invariant conserved current as

$$J^{\Sigma}{}_{\Upsilon} \doteq (-1)^{|\Sigma||\Upsilon|} \bar{\Phi}^S_{\Upsilon} (e^V)^R_S \Phi^{\Sigma}_R, \tag{3.6}$$

where J is a hermitian matrix-valued (indeed, $\mathfrak{u}(2,2|4)$-valued) superfield which is linear:

$$D_+ D_- J^{\Upsilon}{}_{\Lambda} = 0 \quad \text{(on-shell)}. \tag{3.7}$$

Due to the V equation of motion (2.15), the super-trace of this $\mathfrak{u}(2,2|4)$ current gives the diameter (squared)

$$(-1)^{|\Sigma|} J^{\Sigma}{}_{\Sigma} = 4t \tag{3.8}$$

of the Grassmannian manifold. Finally, the conserved charge is

$$Q^{\Sigma}{}_{\Upsilon} = \int d\sigma \left[\int d\kappa^+ d\bar{\kappa}^+ J^{\Sigma}{}_{\Upsilon} + \int d\kappa^- d\bar{\kappa}^- J^{\Sigma}{}_{\Upsilon} \right]. \tag{3.9}$$

The vector components, given by

$$(J_{\ddagger})^{\Sigma}{}_{\Upsilon} \doteq [D_+, \bar{D}_+] J^{\Sigma}{}_{\Upsilon}, \quad (J_{=})^{\Sigma}{}_{\Upsilon} \doteq [D_-, \bar{D}_-] J^{\Sigma}{}_{\Upsilon}. \tag{3.10}$$

can be used to write this charge simply as $Q^{\Sigma}{}_{\Upsilon} = \int d\sigma J^{\Sigma}_{\tau}{}_{\Upsilon}$. In this formula, and all such formulæ for charges appearing henceforth, we take only the lowest component of each superfield on the right-hand-side of the equation.

Since the worldsheet spinors prefer lightcone coordinates, it is convenient for the execution of superspace manipulations to work in this basis. The lightcone time will be taken to be $\sigma^{=} = \frac{1}{2}(\tau - \sigma)$. Then, the lightcone charge is given by $Q_{\text{lc}} = \int \mathrm{d}\sigma^{\ddagger} J_{\ddagger}$ and conservation $\partial_{=} Q_{\text{lc}} = 0$ follows from the identity $i\partial_{\ddagger}[D_-, \bar{D}_-] + i\partial_{=}[D_+, \bar{D}_+] = [D_+ D_-, \bar{D}_+ \bar{D}_-]$ and linearity (3.7) of J.

4 Classical integrability

Besides the global symmetry described above, the action (2.13) admits non-local symmetries. This ought to be true, at least classically, since the gauged linear sigma model is related by a field redefinition to the pure spinor string in $AdS_5 \times S^5$, and the latter has non-local charges [7]. Validity of this description of the pure spinor string in quantum theory requires that these symmetries are not anomalous [8]. The existence of an infinite number of conserved charges is regarded as an indication that the model is integrable. In this section, we will show how the first non-local charge is constructed from the $\mathfrak{u}(2,2|4)$ current J. We then construct the superspace Lax operators generating the complete set of non-local charges and explain the connection to the more familiar component analysis [26].

4.1 Classical non-local charge

An interesting property of the current (3.6) is the identity (valid off-shell when $g \to \infty$)

$$J^{\Sigma}{}_{\Upsilon} J^{\Upsilon}{}_{\Theta} = -t J^{\Sigma}{}_{\Theta}, \tag{4.1}$$

which holds due to equation (2.15) and the definition (3.6). For ease of reference, we will call this equation the "first fundamental J-identity". Although this equation looks like an ordinary algebraic equation, we have to remember that the superfields Φ and $\bar{\Phi}$ are constrained (*viz.* chiral).

We now derive the two remaining identities. Multiplying equation (2.15) on left by Φ^{Υ}_{R} we obtain an off-shell identity which, together with its conjugate, can be written as

$$J^{\Upsilon}{}_{\Sigma} \Phi^{\Sigma}_{S} = -t \Phi^{\Upsilon}_{S} \text{ and } \bar{\Phi}^{S}_{\Sigma} J^{\Sigma}{}_{\Upsilon} = -t \bar{\Phi}^{S}_{\Upsilon}. \tag{4.2}$$

Applying $\bar{D}_{\pm}$ on the first equation and using chirality, one obtains $(\bar{D}_{\pm} J)\Phi = 0$. Taking the complex conjugate of this equation gives $\bar{\Phi}(D_{\pm} J) = 0$. These two equations imply the second and third fundamental identities[8]

$$(\bar{D}_{\pm} J^{\Upsilon}{}_{\Sigma}) J^{\Sigma}{}_{\Lambda} = 0 \text{ and } (-1)^{|\Sigma|} J^{\Upsilon}{}_{\Sigma} (D_{\pm} J^{\Sigma}{}_{\Lambda}) = 0. \tag{4.3}$$

These two equations together with (4.1) will form the basic set of fundamental off-shell equations. They represent the superspace analogue of the flatness condition in two-dimensional classical integrable models. Combined with the on-shell relation (3.7), they are, in fact, equivalent to the V equation of motion, and chirality and equations of motion of Φ, thus providing the necessary ingredients to construct the flat component current and Lax operators.

To write the component equation for the curl of the conserved current, it is useful to define a second component current constructed from fermion bi-linears:

$$\begin{aligned} j^{\Upsilon}_{\ddagger}{}_{\Lambda} &= -\frac{2}{t}(-1)^{|\Upsilon|+|\Sigma|} \left(D_{+} J^{\Upsilon}{}_{\Sigma} \bar{D}_{+} J^{\Sigma}{}_{\Lambda} + \bar{D}_{+} J^{\Upsilon}{}_{\Sigma} D_{+} J^{\Sigma}{}_{\Lambda} \right), \\ j^{\Upsilon}_{=}{}_{\Lambda} &= -\frac{2}{t}(-1)^{|\Upsilon|+|\Sigma|} \left(D_{-} J^{\Upsilon}{}_{\Sigma} \bar{D}_{-} J^{\Sigma}{}_{\Lambda} + \bar{D}_{-} J^{\Upsilon}{}_{\Sigma} D_{-} J^{\Sigma}{}_{\Lambda} \right). \end{aligned} \tag{4.4}$$

In appendix C we show that these currents, together with $J_{\ddagger,=}$ satisfy the "flatness equation" [26]

$$it\partial_{\ddagger} \left(J^{\Upsilon}_{=}{}_{\Lambda} + j^{\Upsilon}_{=}{}_{\Lambda} \right) - it\partial_{=} \left(J^{\Upsilon}_{\ddagger}{}_{\Lambda} + j^{\Upsilon}_{\ddagger}{}_{\Lambda} \right) + \left[J^{\Upsilon}_{\ddagger}{}_{\Sigma} J^{\Sigma}_{=}{}_{\Lambda} - J^{\Upsilon}_{=}{}_{\Sigma} J^{\Sigma}_{\ddagger}{}_{\Lambda} \right] = 0\ . \tag{4.5}$$

With this, we are able to write down the non-local charge. In lightcone coordinates,

$$\mathcal{Q}_{\rm lc} = \iint \mathrm{d}\sigma_1^{\ddagger} \mathrm{d}\sigma_2^{\ddagger} \theta(\sigma_1^{\ddagger} - \sigma_2^{\ddagger}) [J_{\ddagger}(\sigma_1), J_{\ddagger}(\sigma_2)] + 2it \int \mathrm{d}\sigma^{\ddagger} (J_{\ddagger} + j_{\ddagger}). \tag{4.6}$$

Using (4.5) and the conservation of the vector components of the currents, it is straightforward to verify that

$$\partial_{=} \mathcal{Q}_{\rm lc} = 0, \tag{4.7}$$

that is, the non-local charge is conserved.

[8] Although we give them different names for easy of reference, the third identity is the hermitian conjugate of the second.

4.2 The superspace Lax "quartet"

In this section we construct a superspace Lax representation of the flatness equation. The starting point is to construct the following "pure gauge connections"

$$\mathbf{D}_{+}{}^{\Upsilon}{}_{\Lambda} = (-1)^{|\Sigma|}(e^{-\frac{\lambda}{t}J})^{\Upsilon}{}_{\Sigma}D_{+}\,(e^{\frac{\lambda}{t}J})^{\Sigma}{}_{\Lambda}, \quad \bar{\mathbf{D}}_{+}{}^{\Upsilon}{}_{\Lambda} = (-1)^{|\Sigma|}(e^{\frac{\lambda}{t}J})^{\Upsilon}{}_{\Sigma}\bar{D}_{+}\,(e^{-\frac{\lambda}{t}J})^{\Sigma}{}_{\Lambda},$$
$$\mathbf{D}_{-}{}^{\Upsilon}{}_{\Lambda} = (-1)^{|\Sigma|}(e^{\frac{\lambda}{t}J})^{\Upsilon}{}_{\Sigma}D_{-}\,(e^{-\frac{\lambda}{t}J})^{\Sigma}{}_{\Lambda}, \quad \bar{\mathbf{D}}_{-}{}^{\Upsilon}{}_{\Lambda} = (-1)^{|\Sigma|}(e^{-\frac{\lambda}{t}J})^{\Upsilon}{}_{\Sigma}\bar{D}_{-}\,(e^{+\frac{\lambda}{t}J})^{\Sigma}{}_{\Lambda}. \tag{4.8}$$

From these definitions it is clear that $\{\mathbf{D}_{+},\bar{\mathbf{D}}_{-}\}=\{\mathbf{D}_{-},\bar{\mathbf{D}}_{+}\}=0$. To check which other supercurvatures are zero we expand the derivations (4.8). Due to the first fundamental equation (4.1), it is easy to compute that

$$(e^{\frac{\lambda}{t}J})^{\Sigma}{}_{\Lambda} = \delta^{\Sigma}{}_{\Lambda} + \frac{1}{t}(1-e^{-\lambda})J^{\Sigma}{}_{\Lambda} \tag{4.9}$$

and, therefore, the derivations are at most quadratic in J. We now show that they are, in fact, linear. Explicitly we have

$$\mathbf{D}_{+}{}^{\Sigma}{}_{\Theta} = (-1)^{|\Sigma|}\delta^{\Sigma}{}_{\Theta}D_{+} + \frac{1}{t}(-1)^{|\Lambda|}(1-e^{-\lambda})(\delta^{\Sigma}{}_{\Lambda} + \frac{1}{t}(1-e^{\lambda})J^{\Sigma}{}_{\Lambda})D_{+}J^{\Lambda}{}_{\Theta}. \tag{4.10}$$

We can simplify this by using the third fundamental relation (4.3) to obtain

$$\mathbf{D}_{+}{}^{\Sigma}{}_{\Theta} = (-1)^{|\Sigma|}\left(\delta^{\Sigma}{}_{\Theta}D_{+} + \frac{1}{t}(1-e^{-\lambda})D_{+}J^{\Sigma}{}_{\Theta}\right). \tag{4.11}$$

The construction of $\bar{\mathbf{D}}_{+}$ involves an additional step: At first we have

$$\begin{aligned}\bar{\mathbf{D}}_{+}{}^{\Sigma}{}_{\Theta} &= (-1)^{|\Sigma|}\delta^{\Sigma}{}_{\Theta}\bar{D}_{+} + \frac{1}{t}(-1)^{|\Lambda|}(1-e^{\lambda})(\delta^{\Sigma}{}_{\Lambda} + \frac{1}{t}(1-e^{-\lambda})J^{\Sigma}{}_{\Lambda})\bar{D}_{+}J^{\Lambda}{}_{\Theta} \qquad (4.12)\\ &= (-1)^{|\Sigma|}\delta^{\Sigma}{}_{\Theta}\bar{D}_{+} + \frac{1}{t}(-1)^{|\Sigma|}(1-e^{\lambda})\bar{D}_{+}J^{\Sigma}{}_{\Theta}\\ &\quad + \frac{1}{t^2}(2-e^{\lambda}-e^{-\lambda})\left((-1)^{|\Lambda|}J^{\Sigma}{}_{\Lambda}\bar{D}_{+}J^{\Lambda}{}_{\Theta}\right).\end{aligned}$$

Now we simplify this using the second fundamental relation (4.3) and the equation $(-1)^{|\Lambda|}J^{\Sigma}{}_{\Lambda}\ \bar{D}_{\pm}J^{\Lambda}{}_{\Theta} = -t(-1)^{|\Sigma|}\bar{D}_{\pm}J^{\Sigma}{}_{\Theta}$, which follows from the first fundamental relation and is derived (c.f. equation C.4) in appendix C. This gives

$$\bar{\mathbf{D}}_{+}{}^{\Sigma}{}_{\Theta} = (-1)^{|\Sigma|}\left(\delta^{\Sigma}{}_{\Theta}\bar{D}_{+} + \frac{1}{t}(e^{-\lambda}-1)\bar{D}_{+}J^{\Sigma}{}_{\Theta}\right). \tag{4.13}$$

Along exactly the same lines, the remaining two derivations are giving by

$$\begin{aligned}\mathbf{D}_{-}{}^{\Sigma}{}_{\Theta} &= (-1)^{|\Sigma|}\left(\delta^{\Sigma}{}_{\Theta}D_{-} + \frac{1}{t}(1-e^{\lambda})D_{-}J^{\Sigma}{}_{\Theta}\right),\\ \bar{\mathbf{D}}_{-}{}^{\Sigma}{}_{\Theta} &= (-1)^{|\Sigma|}\left(\delta^{\Sigma}{}_{\Theta}D_{-} + \frac{1}{t}(e^{\lambda}-1)\bar{D}_{-}J^{\Sigma}{}_{\Theta}\right).\end{aligned} \tag{4.14}$$

It is now easy to show that the supercurvature $\{\mathbf{D}_{+},\mathbf{D}_{-}\}$ vanishes if and only if all on- and off-shell J-identities hold. The same is true for $\{\bar{\mathbf{D}}_{+},\bar{\mathbf{D}}_{-}\}$. With these four derivations we define the compatible system of equations

$$\mathbf{D}_{\pm}\mathrm{U}(\sigma^{\ddagger},\sigma^{=}\,;\lambda)=0 \quad \text{or} \quad \bar{\mathbf{D}}_{\pm}V(\sigma^{\ddagger},\sigma^{=}\,;\lambda)=0 \tag{4.15}$$

whose solutions generate infinitely many conservation laws. In order to make contact with the usual Lax pair construction in bosonic integrable models we have to compute the two remaining supercurvatures. We will define them as

$$\mathbf{D}_{\ddagger} = \frac{i}{2}\{\mathbf{D}_+, \bar{\mathbf{D}}_+\} \quad \text{and} \quad \mathbf{D}_{=} = \frac{i}{2}\{\mathbf{D}_-, \bar{\mathbf{D}}_-\}. \tag{4.16}$$

Their explicit expressions can be computed using the equations above and the result is

$$\begin{aligned} \mathbf{D}_{\ddagger}{}^{\Sigma}{}_{\Theta} &= \delta^{\Sigma}{}_{\Theta}\, \partial_{\ddagger} + \frac{1}{2it}(1-e^{-\lambda})\, J_{\ddagger}{}^{\Sigma}{}_{\Theta} - \frac{1}{4it}(1-2e^{-\lambda}+e^{-2\lambda})\, j_{\ddagger}{}^{\Sigma}{}_{\Theta}, \\ \mathbf{D}_{=}{}^{\Sigma}{}_{\Theta} &= \delta^{\Sigma}{}_{\Theta}\, \partial_{=} + \frac{1}{2it}(1-e^{\lambda})\, J_{=}{}^{\Sigma}{}_{\Theta} - \frac{1}{4it}(1-2e^{\lambda}+e^{2\lambda})\, j_{=}{}^{\Sigma}{}_{\Theta}, \end{aligned} \tag{4.17}$$

where $(J_{\ddagger}, J_{=})$ and $(j_{\ddagger}, j_{=})$ were defined in equations (3.10) and (4.4), respectively. Using the vanishing supercurvatures $\{\mathbf{D}_+, \mathbf{D}_-\}$ and $\{\bar{\mathbf{D}}_+, \bar{\mathbf{D}}_-\}$, we automatically have that

$$\mathbf{F}_{\ddagger=}(\lambda) \doteq [\mathbf{D}_{\ddagger}, \mathbf{D}_{=}] = 0 \tag{4.18}$$

which is the equation satisfied by the usual bosonic Lax pair.

Expanding this expression in exponentials of the spectral parameter, we find linearly independent combinations of J-flatness (4.5) and J-conservation, and analogous equations expressing the non-flatness and non-convervation of j. These formulæ are equivalent to those found by component analysis in reference [26] and the derivations in (4.17) correspond precisely to the usual Lax pair in sigma models on Grassmannian manifolds. It is known that solutions of

$$\mathbf{D}_{\ddagger}\mathrm{U}(\sigma^{\ddagger}, \sigma^{=}; \lambda) = \mathbf{D}_{=}\mathrm{U}(\sigma^{\ddagger}, \sigma^{=}; \lambda) = 0 \tag{4.19}$$

lead to infinitely many conservation laws [32]. Of course every solution of (4.15) with $V = U$ is also a solution of (4.19). Whether the reverse is true we leave as an interesting open question. Others aspects of integrable supersymmetric sigma models can be found at [27, 28]

5 Quantum integrability

The computations in section 4.1 relevant to the definition of the non-local charge (4.6) are a mixture of superspace and component calculations. To study the quantum analogue of the conservation of the non-local charge, one can proceed with the component analysis along the lines of reference [29]. However, the non-local term in the charge is most easily proven to be unrenormalized by embedding it in superspace. We therefore prefer to keep supersymmetry manifest. Furthermore, since the worldsheet fermions κ prefer lightcone coordinates, we will perform all calculations in this section in the lightcone basis.

5.1 Embedding of the non-local charge in superspace

We begin by proposing an $N = (2, 2)$ generalization of the Heaviside function. This will be the formal substitution of the worldsheet supercoordinate in the ordinary Heaviside function. To construct the appropriate worldsheet supercoordinate we start with the chiral representation superspace lightcone coordinates $\sigma_1^{\ddagger} - \sigma_2^{\ddagger} + i\bar\kappa_2^+\kappa_1^+$ and $\sigma_1^{=} - \sigma_2^{=} + i\bar\kappa_2^-\kappa_1^-$. These coordinates have the property that they are annihilated by $\bar D_{\pm 1}$ and $D_{\pm 2}$ in the chiral representation. Since we will be working with hermitian superfields, it is appropriate to switch to the real representation[9] obtained by acting with $\mathrm{e}^{i(\kappa\sigma^a\bar\kappa)\partial_a}$. This gives

$$\begin{aligned}\hat\sigma_{12}^{\ddagger} &= \sigma_1^{\ddagger} - \sigma_2^{\ddagger} + i\bar\kappa_2^+\kappa_1^+ + i(\kappa_1^+\bar\kappa_1^+ - \kappa_2^+\bar\kappa_2^+),\\ \hat\sigma_{12}^{=} &= \sigma_1^{=} - \sigma_2^{=} + i\bar\kappa_2^-\kappa_1^- + i(\kappa_1^-\bar\kappa_1^- - \kappa_2^-\bar\kappa_2^-).\end{aligned} \tag{5.1}$$

With this expression for the worldsheet coordinate, the proposal for the Heaviside function is simply

$$\Theta(\sigma_{12}^{\ddagger}) \doteq \theta(\hat\sigma_{12}^{\ddagger}). \tag{5.2}$$

The naïve guess for the first term in the supercharge is the Lorentz covariant integral

$$I_0 = \iint \mathrm{d}\mu_1\mathrm{d}\mu_2\, \Theta(\sigma_{12}^{\ddagger})\, [J(\sigma_1), J(\sigma_2)] \tag{5.3}$$

with the "measure" $\mathrm{d}\mu = \mathrm{d}\sigma[D_+, \bar D_+]|$. To check this we must compute the component projection. To do that it is useful to notice that the Heaviside function depends only on even powers of κ. This, together with anti-symmetry of the commutator, results in only two non-vanishing contributions: One in which both commutators hit the Heaviside function and one in which neither of them do. Direct calculation results in

$$I_0 = \iint \mathrm{d}\sigma_1^{\ddagger}\mathrm{d}\sigma_2^{\ddagger} \left\{\theta(\sigma_{12}^{\ddagger})\, [J_{\ddagger}(\sigma_1), J_{\ddagger}(\sigma_2)] + 4\delta'(\sigma_{12}^{\ddagger})\, [J(\sigma_1), J(\sigma_2)]\right\}. \tag{5.4}$$

Integrating the δ' term over $\sigma_{1,2}$ we get

$$-4 \iint \mathrm{d}\sigma_1^{\ddagger}\mathrm{d}\sigma_2^{\ddagger}\, \delta(\sigma_1^{\ddagger} - \sigma_2^{\ddagger})\, [\partial_{\ddagger} J(\sigma_1), J(\sigma_2)] = 4 \int \mathrm{d}\sigma J \overleftrightarrow{\partial_{\ddagger}} J. \tag{5.5}$$

Although this type of term does not look familiar, we show in appendix C that

$$J \overleftrightarrow{\partial_a} J = \frac{it}{2}(j_a - J_a), \tag{5.6}$$

where a can be any of the indices $\ddagger, =, \tau, \sigma$. The superspace integral is therefore expressible as

$$I_0 = \iint \mathrm{d}\sigma_1^{\ddagger}\mathrm{d}\sigma_2^{\ddagger}\, \theta(\sigma_{12}^{\ddagger})\, [J_{\ddagger}(\sigma_1), J_{\ddagger}(\sigma_2)] + 2it \int \mathrm{d}\sigma^{\ddagger} \left(-J_{\ddagger} + j_{\ddagger}\right). \tag{5.7}$$

It follows that the non-local charge in lightcone coordinates is expressible entirely in terms of a the $\mathfrak{u}(2,2|4)$ supercurrent as

$$\mathfrak{Q}_{\mathrm{lc}} = \iint \mathrm{d}\mu_1\mathrm{d}\mu_2\, \Theta(\sigma_{12}^{\ddagger})\, [J(\sigma_1), J(\sigma_2)] + 4it \int \mathrm{d}\mu\, J. \tag{5.8}$$

The precise relative coefficient in the component expression (5.4) is crucial to match the coefficient of $j_{\ddagger}$ in (4.6). This is important since, contrary to $J_{\ddagger} = [D_+, \bar D_+]J$, it is impossible to write $j_{\ddagger}$ as an expression of the form $\big($(combination of D_+ and $\bar D_+$) acting on (function of J)$\big)$. It would have followed from this that there is no superspace expression, the lowest component of which is the non-local charge.

[9] We thank Martin Roček and Warren Siegel for reminding us of this expression.

5.2 Non-renormalization

We now examine the possible renormalization of the the non-local $\mathfrak{u}(2,2|4)$ charge. The superspace form (5.8) shows that if the charge is renormalized, it will happen due to the operator product of the $\mathfrak{u}(2,2|4)$ currents $J(\sigma_1)$ and $J(\sigma_2)$. We will now show that the supergroup nature of this operator product cancels this potential divergence.

Consider, again, the $\mathfrak{u}(2,2|4)$ current $J^{\Upsilon}{}_{\Sigma}$. Irrespective of how it is defined, its operator product expansion with any operator $\mathcal{O}^{\Theta}$ in the fundamental representation is on general grounds

$$J^{\Upsilon}{}_{\Sigma}(\hat{\sigma}_1)\mathcal{O}^{\Theta}(\hat{\sigma}_2) \sim -\log|\hat{\sigma}_{12}|^2\, T^{\Upsilon\Theta}_{\Sigma\Gamma}\, \mathcal{O}^{\Gamma}(\hat{\sigma}_{1+2}) \tag{5.9}$$

for some $\mathfrak{u}(2,2|4)$-invariant tensor T. Where we assumed that $\mathcal{O}^{\Theta}$ is a chiral operator. This OPE is constrained by the fact that $\mathcal{O}^{\Theta}$ must transform under a global $\mathrm{U}(2,2|4)$ transformation as

$$[M^{\Sigma}{}_{\Upsilon}Q^{\Upsilon}{}_{\Sigma}\,, \mathcal{O}^{\Theta}] = M^{\Theta}{}_{\Upsilon}\mathcal{O}^{\Upsilon}, \tag{5.10}$$

where $Q^{\Upsilon}{}_{\Sigma}$ is given by (3.9). This form is also fixed by the classical weight of J, which does not change in the quantum theory since J is a conserved current. The tensor structure is determined by the action of $\mathfrak{u}(2,2|4)$. In appendix B we review the construction of the $\mathfrak{u}(2,2|4)$ algebra. There we find that $T^{\Upsilon\Theta}_{\Sigma\Gamma} = (-1)^{|\Theta|}\delta^{\Theta}{}_{\Sigma}\delta^{\Upsilon}{}_{\Gamma}$ is the sign factor (B.4).

The JJ operator product now follows from the adjoint action and also by acting J twice in (5.9)

$$\begin{aligned} J^{\Upsilon}{}_{\Sigma}(\hat{\sigma}_1)J^{\Theta}{}_{\Phi}(\hat{\sigma}_2) \sim & \\ -(\log|\hat{\sigma}_{12}|^2 + \log|\hat{\sigma}_{21}|^2)\Big[&(-1)^{|\Phi|(|\Sigma|+|\Theta|)+|\Sigma||\Theta|}\delta^{\Upsilon}{}_{\Phi}J^{\Theta}{}_{\Sigma}(\hat{\sigma}_{1+2}) \\ &-(-1)^{|\Sigma||\Theta|}\delta^{\Theta}{}_{\Sigma}J^{\Upsilon}{}_{\Phi}(\hat{\sigma}_{1+2})\Big] \end{aligned} \tag{5.11}$$

where we have included $\log|\hat{\sigma}_{21}|^2$ so that this OPE is hermitian. Note that $\hat{\sigma}_{12}$ is *not* antisymmetric in 1 and 2.[10] What enters the quantum charge, however, is the matrix product. This corresponds to summing over $\Theta = \Sigma$. This gives (c.f. equation (B.6))

$$J^{\Upsilon}{}_{\Sigma}(\hat{\sigma}_1)J^{\Sigma}{}_{\Phi}(\hat{\sigma}_2) \sim -4t(\log|\hat{\sigma}_{12}|^2 + \log|\hat{\sigma}_{21}|^2)\delta^{\Upsilon}{}_{\Phi}, \tag{5.12}$$

where we have used equation (3.8) for the super-trace of J. Finally, what enters the quantum non-local charge is the commutator

$$J^{\Upsilon}{}_{\Sigma}(\hat{\sigma}_1)J^{\Sigma}{}_{\Phi}(\hat{\sigma}_2) - J^{\Upsilon}{}_{\Sigma}(\hat{\sigma}_2)J^{\Sigma}{}_{\Phi}(\hat{\sigma}_1) \sim 0 \tag{5.13}$$

which is, therefore, not renormalized. All other potential quantum corrections to the equation above are of order $|\sigma|^2$ since the gauge coupling constant and other gauge invariant operators have negative length dimension.

One can calculate from this OPE the corresponding OPEs of the vector components of J, and they all vanish. This result has two consequences. First, it means that the classical

[10] Terms like $\frac{\sigma^{\pm}}{\sigma^{\mp}}$ are also forbidden in these OPEs since J is a worldsheet scalar.

non-local charge (4.6) is well defined in the quantum theory. Also, as a quantum operator, it is conserved since all equations needed to prove this do not receive quantum corrections.

We have explicitly worked out the details of the operator products for the $\mathfrak{u}(2,2|4)$ current of the $\mathrm{U}(2,2|4)/\mathrm{U}(2,2) \times \mathrm{U}(4)$ and shown that the non-local charge constructed from this current is not renormalized. This result holds in more generality. Let us replace $\mathrm{U}(2,2|4)$ with a general supergroup G with Lie super-algebra $\mathfrak{g}$. Let $H \subset G$ be a subgroup and $K \subset G$ its commutant. The gauged linear sigma model on the Grassmannian manifold $G/(K \times H)$ can be constructed along the lines section 2, the H-invariant current as in section 3, the non-local G-charge by the results of section 4, and finally, its embedding in superspace performed in this section. What is then required is to repeat the steps considered here to show that the generator of this non-local symmetry is not renormalized if the last term on the right-hand-side of (5.11) vanishes. The operator product expansions entering this calculation are, again, fixed by conformal weights and the representation theory of $\mathfrak{g}$. Since we use only the fundamental and adjoint representations, equations analogous to (5.10) and (5.11) hold. In the final step we take the matrix product of the currents. The coefficient of the resulting operator product is simply the generalization of the dual Coxeter number to the Lie super-algebra $\mathfrak{g}$. We, therefore, conclude that any $N = (2,2)$ non-linear sigma model with Grassmannian target manifold constructed from a supergroup G with vanishing dual Coxeter number has a non-local G-symmetry which is protected from renormalization.

In the case of no supersymmetry on the worldsheet, the Grassmannian sigma model has an anomaly (i.e. a gauge field strength appearing on the right-hand-side of (5.11)) which prevents the non-local charge from being conserved [26, 33]. This anomaly disappears on the $N = (1,1)$ supersymmetric worldsheet which can also be seen from the fact that the dimension of the supersymmetric field strength prevents it from appearing in the OPE of the supercurrents.

The renormalization of the non-local charge in the case of non-vanishing dual Coxeter number is intimately related the the existence of a mass gap in the theory. Since the Berkovits-Vafa GSLM does not have a mass gap [30] it is natural to find that the non-local charge is not renormalized.

6 Conclusions and further directions

In this paper we analyzed the classical and quantum integrability properties of the gauged linear sigma model proposed for the pure spinor superstring in $AdS_5 \times S^5$ background by Berkovits and Vafa [9]. A superspace non-local charge was constructed and was proven to be conserved at both the classical and quantum level. Furthermore, we constructed a superspace Lax "quartet", which could be used to study the integrability of the model directly in superspace. However, the use of such nontrivial conservation laws in the present model still remains to be uncovered.

There are many interesting directions which deserve further study. One outstanding problem is to understand the precise mapping between physical deformations of the original pure spinor action and physical deformations of the action (2.13). As we noted in the introduction and in section 2 the mapping will break worldsheet supersymmetry since the cohomology is not defined as that of the usual A-models. It would be interesting to see whether the non-local charge constructed here commutes with the pure spinor BRST charge, as in [34].

Another important open problem is a careful analysis of the $t \to 0$ limit. We can see from the results above that the present approach fails in this limit, since the construction does not work in this case (many expressions are singular for $t = 0$). Moreover, in this limit, one cannot eliminate the gauge degrees of freedom. It is reasonable to expect that a suitable combination of limits of both coupling constants leads to the existence of some other nontrivial conservation laws.

In [35] the spectra of some coset sigma models with target space supersymmetry were computed. These sigma models can be thought of as supersymmetric generalizations of the $\vec{n}$ field model with a suitable topological term turned on, which in the present case, means a non-zero θ-angle in the superpotential (2.12). One might wonder whether similar methods could be generalized for symmetric space cosets.

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A The U(2, 2|4) supergroup

The supergroup U(2, 2|4) can be thought as the group of unitary transformation of an eight-dimensional vector space where the first four components are usual complex numbers and the remaining four are complex Grassmann numbers. For example, let us denote X^Σ and element of this vector space. The index Σ splits into $A = 1 \cdots 4$ and $J = 1 \cdots 4$, i.e. $X^\Sigma = (x^A, \theta^J)$. Here, x^A are complex numbers and θ^J are complex Grassmann numbers. The metric in this space is $\eta_{\bar{\Sigma}\Upsilon} = (\eta_{\bar{A}B}, \eta_{\bar{I}J})$, with $\eta_{\bar{A}J} = \eta_{\bar{I}B} = 0$. Furthermore, $\eta_{\bar{A}B} = \mathrm{diag}(1, 1, -1, -1)$ and $\eta_{\bar{I}J} = \mathrm{diag}(1, 1, 1, 1)$. The elements of U(2, 2|4) preserve the inner product

$$\bar{Y}^{\bar{\Sigma}} \eta_{\bar{\Sigma}\Upsilon} X^\Upsilon = \bar{Y}_\Sigma X^\Sigma = (\bar{Y}')_\Sigma (X')^\Sigma, \tag{A.1}$$

where $(X')^\Sigma = M^\Sigma{}_\Omega X^\Omega$, $(\bar{Y}')_\Sigma = \bar{Y}_\Omega (M^\dagger)^\Omega{}_\Sigma$ and $M^\Sigma{}_\Omega$ is an element of U(2, 2|4). Note that the supermatrix M has the following form

$$M^\Sigma{}_\Omega = \begin{pmatrix} m^A{}_B & f^A{}_J \\ g^I{}_B & n^I{}_J \end{pmatrix}, \tag{A.2}$$

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where m and n are usual complex matrices and f and g are Grassmann valued matrices. The conditions from invariance of the inner product impose on these matrices are

$$
\begin{aligned}
(M^\dagger)^\Omega{}_\Sigma M^\Sigma{}_\Upsilon &= \begin{pmatrix} (m^\dagger)^A{}_B m^B{}_C + (g^\dagger)^A{}_J g^J{}_C & (m^\dagger)^A{}_B f^B{}_K + (g^\dagger)^A{}_J n^J{}_K \\ \\ (f^\dagger)^I{}_B m^B{}_C + (n^\dagger)^I{}_J g^J{}_C & (f^\dagger)^I{}_B f^B{}_K + (n^\dagger)^I{}_J n^J{}_K \end{pmatrix} \\
&= \begin{pmatrix} \eta^A{}_C & 0 \\ \\ 0 & \eta^I{}_K \end{pmatrix}.
\end{aligned} \tag{A.3}
$$

These conditions can be solved factorizing M into two matrices

$$
M^\Sigma{}_\Omega = T^\Sigma{}_\Upsilon U^\Upsilon{}_\Omega, \tag{A.4}
$$

where the matrices U and T are given by

$$
U^\Sigma{}_\Omega = \begin{pmatrix} u^A{}_B & 0 \\ 0 & v^I{}_J \end{pmatrix}, \tag{A.5}
$$

$$
T^\Sigma{}_\Omega = \begin{pmatrix} \left(\frac{1}{\sqrt{1+ZZ^\dagger}}\right)^A{}_B & Z^A{}_J \left(\frac{1}{\sqrt{1+Z^\dagger Z}}\right)^J{}_K \\ \\ \left(\frac{1}{\sqrt{1+Z^\dagger Z}}\right)^I{}_J (Z^\dagger)^J{}_B & \left(\frac{1}{\sqrt{1+Z^\dagger Z}}\right)^I{}_K \end{pmatrix}, \tag{A.6}
$$

where u and v are two arbitrary U(2, 2) and U(4) matrices respectively and Z is an arbitrary complex Grassmann valued matrix.

B The $\mathfrak{u}(2, 2|4)$ algebra

Here we describe the superalgebra $\mathfrak{u}(2, 2|4) \cong \mathfrak{gl}(4|4)$. We use mostly the definitions and conventions of [36, 37]. In order to describe this algebra, we introduce the set of oscillators $\mathbf{A}^\Sigma = (\mathbf{a}^\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^J)$, where we have split the A index into $(\alpha, \dot\alpha)$. Also, we define the hermitian conjugate to be $\mathbf{A}^\dagger_\Sigma = (\mathbf{a}^\dagger_\alpha, -\mathbf{b}^{\dot\beta}, \mathbf{c}^\dagger_J)$. The oscillators $(\mathbf{a}, \mathbf{b})$ are bosonic and the oscillators $\mathbf{c}$ are fermionic. They satisfy the following (anti-)commutation relations:

$$
[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] = \delta^\alpha_\beta, \quad [\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}, \quad \{\mathbf{c}^J, \mathbf{c}^\dagger_K\} = \delta^J_K, \tag{B.1}
$$

so we can define the graded commutators of $\mathbf{A}$ and $\mathbf{A}^\dagger$ as

$$
[\mathbf{A}^\Sigma, \mathbf{A}^\dagger_\Upsilon] \doteq \mathbf{A}^\Sigma \mathbf{A}^\dagger_\Upsilon - (-1)^{|\Sigma||\Upsilon|} \mathbf{A}^\dagger_\Upsilon \mathbf{A}^\Sigma = \delta^\Sigma_\Upsilon, \tag{B.2}
$$

where $|\Sigma| = 0, 1$ is the grading of the corresponding mode of the oscillator. Using the above definitions, the generators of $\mathfrak{u}(2, 2|4)$ are written as

$$
\mathfrak{J}^\Sigma{}_\Upsilon \doteq (-1)^{|\Sigma||\Upsilon|} \mathbf{A}^\dagger_\Upsilon \mathbf{A}^\Sigma. \tag{B.3}
$$

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Note that the supervector $\mathbf{A}$ forms a fundamental representation of $\mathfrak{u}(2,2|4)$ in the sense that

$$[\mathfrak{J}^{\Sigma}{}_{\Upsilon}, \mathbf{A}^{\Theta}] = -(-1)^{|\Theta|}\delta^{\Theta}_{\Upsilon}\mathbf{A}^{\Sigma}. \tag{B.4}$$

The (anti-)commutation relations of the generators can be easily computed using (B.2)

$$[\mathfrak{J}^{\Sigma}{}_{\Upsilon}, \mathfrak{J}^{\Theta}{}_{\Lambda}] = (-1)^{|\Lambda|(|\Theta|+|\Upsilon|)+|\Theta||\Upsilon|}\delta^{\Sigma}_{\Lambda}\mathfrak{J}^{\Theta}{}_{\Upsilon} - (-1)^{|\Upsilon|}\delta^{\Theta}{}_{\Upsilon}\mathfrak{J}^{\Sigma}{}_{\Lambda}. \tag{B.5}$$

It is interesting to compute the above commutator with the indices Υ and Θ contracted

$$[\mathfrak{J}^{\Sigma}{}_{\Theta}, \mathfrak{J}^{\Theta}{}_{\Lambda}] = (-1)^{|\Theta|}\delta^{\Sigma}_{\Lambda}\mathfrak{J}^{\Theta}{}_{\Theta} - \mathfrak{J}^{\Sigma}{}_{\Lambda}((-1)^{|\Theta|}\delta^{\Theta}{}_{\Theta}) = -2\delta^{\Sigma}_{\Lambda}\mathfrak{C}, \tag{B.6}$$

where $\mathfrak{C} = -\frac{1}{2}(-1)^{|\Theta|}\mathfrak{J}^{\Theta}{}_{\Theta} = -\frac{1}{2}\mathbf{A}^{\dagger}_{\Theta}\mathbf{A}^{\Theta}$ is the central charge operator. The other possible trace of the generators is $\mathfrak{J}^{\Theta}{}_{\Theta} = (-1)^{|\Theta|}\mathbf{A}^{\dagger}_{\Theta}\mathbf{A}^{\Theta} = 2\mathfrak{C} + 4\mathfrak{B}$ where

$$\mathfrak{B} = \frac{1}{4}\mathfrak{J}^{\Theta}{}_{\Theta} + \frac{1}{4}(-1)^{|\Theta|}\mathfrak{J}^{\Theta}{}_{\Theta} \tag{B.7}$$

is the hypercharge [37]. These two traced generators can be removed from the $\mathfrak{u}(2,2|4)$ algebra, and the end result is the $\mathfrak{psu}(2,2|4)$ algebra.

C Derivation of the "Flatness equation" and "J-relation"

In this section we derive the flatness equation (4.5) used to construct the non-local conserved charge $\mathcal{Q}$ (4.6) and the relation (5.6) between J, J_a, and j_a. As this will involve some superspace gymnastics, we introduce the following notational aid for the commutator of two superspace derivatives: $\triangle_{\alpha\dot{\alpha}} \doteq [D_\alpha, \bar{D}_{\dot{\alpha}}]$. The dotted index refers to a label on a conjugated superspace derivative.

Hitting the second and third fundamental identities (4.3) with D and $\bar{D}$ we find the relations

$$\begin{aligned} &-i\partial_{\alpha\dot{\alpha}}J^{\Upsilon}{}_{\Sigma}J^{\Sigma}{}_{\Upsilon} + \frac{1}{2}\triangle_{\alpha\dot{\alpha}}J^{\Upsilon}{}_{\Sigma}J^{\Sigma}{}_{\Lambda} - (-1)^{|\Upsilon|+|\Sigma|}\bar{D}_{\dot{\alpha}}J^{\Upsilon}{}_{\Sigma}D_{\alpha}J^{\Sigma}{}_{\Lambda} = 0, \\ &-iJ^{\Upsilon}{}_{\Sigma}\partial_{\alpha\dot{\alpha}}J^{\Sigma}{}_{\Upsilon} - \frac{1}{2}J^{\Upsilon}{}_{\Sigma}\triangle_{\alpha\dot{\alpha}}J^{\Sigma}{}_{\Lambda} + (-1)^{|\Upsilon|+|\Sigma|}\bar{D}_{\dot{\alpha}}J^{\Upsilon}{}_{\Sigma}D_{\alpha}J^{\Sigma}{}_{\Lambda} = 0, \end{aligned} \tag{C.1}$$

where we have temporarily resorted to four-dimensional spinor notation to avoid a proliferation of formulæ. Summing these equations and using the first fundamental identity (4.1) yields

$$it\partial_a J - \frac{1}{2}J \overleftrightarrow{\triangle}_a J = 0 \tag{C.2}$$

where $a = \ddagger, =$ or $a = \tau, \sigma$. Taking the difference, we find

$$iJ^{\Upsilon}{}_{\Sigma} \overleftrightarrow{\partial}_{\alpha\dot{\alpha}} J^{\Sigma}{}_{\Lambda} - \frac{t}{2}\triangle_{\alpha\dot{\alpha}}J^{\Upsilon}{}_{\Lambda} - 2(-1)^{|\Upsilon|+|\Sigma|}\bar{D}_{\dot{\alpha}}J^{\Upsilon}{}_{\Sigma}D_{\alpha}J^{\Sigma}{}_{\Lambda}. \tag{C.3}$$

Next, we rearrange the second and third fundamental identity to give

$$-t\bar{D}_{\dot{\alpha}}J^{\Upsilon}{}_{\Lambda} - (-1)^{|\Upsilon|+|\Sigma|}J^{\Upsilon}{}_{\Sigma}\bar{D}_{\dot{\alpha}}J^{\Sigma}{}_{\Lambda} = 0 \text{ and } -tD_{\alpha}J^{\Upsilon}{}_{\Lambda} - J^{\Upsilon}{}_{\Sigma}D_{\alpha}J^{\Sigma}{}_{\Lambda} = 0. \tag{C.4}$$

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Hitting the first with D and the second with $\bar{D}$ we get some unilluminating equations. The sum of these again gives (C.2) but the difference gives

$$iJ^{\Upsilon}{}_{\Sigma} \overleftrightarrow{\partial}_{\alpha\dot{\alpha}} J^{\Sigma}{}_{\Lambda} - \frac{t}{2} \triangle_{\alpha\dot{\alpha}} J^{\Upsilon}{}_{\Lambda} - 2(-1)^{|\Upsilon|+|\Sigma|} D_{\alpha} J^{\Upsilon}{}_{\Sigma} \bar{D}_{\dot{\alpha}} J^{\Sigma}{}_{\Lambda}. \tag{C.5}$$

Adding this to the intermediate result (C.3) and taking the definition of the bi-linear current (4.4) into account, we obtain the formula (5.6) relating J, J_a, and j_a.

We now turn to the flatness equation (4.5). In this computation we take the formula (C.2) and hit it with $\triangle_{\beta\dot{\beta}}$. We are interested in the case in which $a = \ddagger$ and $b = =$ or *vice versa*. The corresponding Ds anti-commute and terms with 3 Ds can be rewritten using the linearity of J. Taking all of this into account the formula simplifies to

$$\begin{aligned} \triangle_b(it\partial_a J^{\Upsilon}{}_{\Lambda}) = \frac{1}{2}\left(\triangle_b J^{\Upsilon}{}_{\Sigma} \triangle_a J^{\Sigma}{}_{\Lambda} - \triangle_a J^{\Upsilon}{}_{\Sigma} \triangle_b J^{\Sigma}{}_{\Lambda}\right) - it\partial_a j_b^{\Upsilon}{}_{\Lambda} \\ + \frac{1}{2}\left(J^{\Upsilon}{}_{\Sigma} \triangle_b \triangle_a J^{\Sigma}{}_{\Lambda} - \triangle_b \triangle_a J^{\Upsilon}{}_{\Sigma} J^{\Sigma}{}_{\Lambda}\right). \end{aligned} \tag{C.6}$$

Switching a and b and subtracting cancels the second line and gives the desired relation (4.5).

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